JOURNAL OF COMPUTATIONAL PHYSICS 72, 299–310 (1987)

Tau Method Approximation of Differential Eigenvalue Problems where the Spectral Parameter Enters Nonlinearly

K. M. LIU* AND E. L. ORTIZ

Department of Mathematics, Imperial College, London, England

Received December 20, 1985; revised October 24, 1986

We discuss the use of recent new formulations of the Tau method for the numerical approximation of differential eigenvalue problems where the spectral parameter appears nonlinearly. Our approach enables us to translate the differential eigenvalue problem into a generalized algebraic eigenvalue problem, which is formulated by using a standard technique easy to implement in a computer. We consider several examples and report results of high accuracy.

1. INTRODUCTION

The numerical treatment of differential eigenvalue problems where the spectral parameter appears nonlinearly has been discussed, in the context of the Tau method, in an early paper of Chaves and Ortiz [1], published in 1968. These authors used the recursive formulation of the Tau method (see Ortiz [2]); although results of considerable accuracy were reported in that paper, the computer implementation of their algorithms was rather involved. In this paper we use a new approach [3–4] to Ortiz' formulation of the Tau method of Lanczos [5], which leads to algorithms of remarkable simplicity, while retaining the accuracy of earlier results.

They are based on the systematic use of two simple matrices, with nonzero elements on only one line parallel to the main diagonal, used to translate the differential eigenvalue problem into an algebraic generalized eigenvalue problem which is solved with standard software.

By using a technique suggested by Peters and Wilkinson [6] we show that the Tau method can be used efficiently for the numerical treatment of differential eigenvalue problems where the spectral parameter enters nonlinearly in the equation. We apply our technique to three examples, the last of which corresponds to a biharmonic problem define on an infinite strip.

The accuracy of the results obtained in this paper suggests that the recent formulations of the Tau method should be taken into account as useful tools for the numerical treatment of differential eigenvalue problems.

* Present address: Department of Mathematical Studies, Hong Kong Polytechnic, Hung Hom. Kowloon, Hong Kong.

0021-9991/87 $3.00

2. Approximation of Differential Eigenvalue Problems with the Tau Method

Let us consider the two matrices

$$\eta := \begin{vmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \cdots & \cdots & \\ & & & \cdots & \cdots \end{vmatrix} \quad \text{and} \quad \mu := \begin{vmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \cdots & \cdots \\ & & & & \cdots \end{vmatrix}.$$

Let $a_n(x) = \underline{\mathbf{a}}_n \cdot \underline{x}$, where $\underline{\mathbf{a}}_n \in \mathbf{R}^{n+1}$, be a polynomial with coefficients a_{ni}, $i = 0(1)\, n$, and let $\underline{\mathbf{x}} = (1, x, x^2, \ldots)^T$. We consider the class $\mathbf{D}$ of linear differential operators with polynomial coefficients; let $D \in \mathbf{D}$, then D is given by

$$D := \sum_{k=0}^{w} p_k(x)[d^k/dx^k] = \sum_{k=0}^{w} \sum_{i=0}^{g_k} p_{ki} x^i [d^k/dx^k],$$

where w is the order of the differential operator D.

We recall the following result from Ortiz and Samara [3]:

Theorem 1. *If $D \in \mathbf{D}$, then*

$$D a_n(x) := \underline{\mathbf{a}}_n \Pi \underline{\mathbf{x}},$$

where

$$\Pi := \sum_{i=0}^{w} \eta^i p_i(\mu).$$

The Tau method can be easily constructed taking as a starting point the elementary case of *linear* differential (or difference-differential) equations with coefficients which are either *polynomials or rational polynomial functions.* From this class of equations its applicability is extended to cases where the coefficients have a more general nature, including the case of discontinuous coefficients (see Liu and Ortiz [7]). Nonlinear ordinary differential equations are treated with techniques discussed in Ortiz [8], Ortiz and Samara [3], Onumanyi and Ortiz [10] and Ortiz and Pham [11]; nonlinear partial differential equations have been discussed by Ortiz and Samara [9] and Ortiz and Pun [12, 13].

Let us consider the differential eigenvalue problem defined by

$$\begin{aligned} D_\lambda\, y(x) &= 0, \qquad a \leqslant x \leqslant b, \\ (f_j, y) &= 0, \qquad j = 0(1)\, w, \end{aligned} \tag{1}$$

where

$$D_\lambda := D_1 + \lambda D_2;\ D_1, D_2 \in \mathbf{D}$$

and the f_j are point functionals acting on $y(x)$; they stand for the boundary conditions to be satisfied by the solution $y(x)$ of (1).

With (1) we associate the *Tau problem*

$$\begin{aligned} D_\lambda y_n(x) &= H_n(x), \qquad && a \leqslant x \leqslant b, \\ (f_j, y_n) &= 0, && j = 1(1)\, w, \end{aligned} \tag{2}$$

defined by the same differential operator D_λ; $H_n(x)$ is a polynomial of degree n chosen for the exact solution $y_n(x)$ of (2) to be a polynomial of a prescribed degree, trivially related to n (see Ortiz [2] for further details).

$H_n(x)$ is a linear combination of polynomials $v_k(x) \in \mathbf{P}_k$ with free coefficients, called the *tau-parameters* of problem (2):

$$H_n(x) := \tau_{n,1} v_n(x) + \cdots + \tau_{n,w+s} v_{n+1-w-s}(x).$$

The $w+s$ tau-parameters are chosen to adjust the approximate solution to the given w boundary conditions and to satisfy s conditions imposed by the Tau method (see Ortiz [2]). The $v_k(x)$ are usually chosen to be polynomials with prescribed optimal properties on $[a, b]$. Chebyshev polynomials, which have minimum properties in the uniform norm on $[a, b]$ and thus *minimize* in that norm the difference between (1) and its associated Tau problem (2), are a common choice. Legendre polynomials, which have the advantage of being orthogonal in $[a, b]$, are selected when it is desirable to minimize the error in the solution at the end points of $[a, b]$ (see Namasivayam and Ortiz [14] for an analytic discussion of these and other possible choices for the representation of $H_n(x)$).

Let $\underline{v} := \{v_k(x)\} = V\underline{x}$, $k = 0, 1, 2, \ldots$, be a polynomial basis, where V is a lower triangular nonsingular matrix. Let us consider the polynomial $y_n(x) = \underline{\underline{\mathbf{a}}}_n \cdot \underline{\mathbf{x}} := \underline{\underline{\mathbf{c}}}_n \cdot \underline{\mathbf{v}}$. Let $\Pi_\lambda := \Pi_1 + \lambda \Pi_2$, where Π_i, $i = 1, 2$, are the matrices associated by Theorem 1 with $D_i \in \mathbf{D}$, $i = 1, 2$. If D is applied to $y_n(x)$, we find that

$$D_\lambda y_n(x) = \underline{\underline{\mathbf{a}}}_n \Pi_\lambda \underline{\mathbf{x}} = \underline{\underline{\mathbf{c}}}_n \hat{\Pi}_\lambda \underline{\mathbf{v}}, \tag{3}$$

where $\hat{\Pi}_\lambda$ is the conjugate of Π_λ under the similarity transformation defined by V.

Let us apply the functionals f_j to $y_n(x)$,

$$(f_j, y_n) = \sum_{i=0}^{n} a_{ni}(f_j, x^i) := \sum_{i=0}^{n} a_{ni} b_{ji} := \underline{\underline{\mathbf{a}}}_n \cdot \underline{\mathbf{b}}_j,$$

for $j = 1(1)\, w$. Let B be a matrix defined by $B := \underline{\mathbf{b}}_j$, $j = 1(1)\, w$. We can write $\underline{\underline{\mathbf{a}}}_n B$ for $(f_j, y_n(x))$, $j = 1(1)\, w$. If $y_n(x)$ is defined in the basis $\underline{v}$ we write $\underline{\underline{\mathbf{c}}}_n VB$. Therefore, we can write

$$(f_j, y_n(x)) = \underline{\underline{\mathbf{c}}}_n VB. \tag{4}$$

Let $[Z]_n$ be the restriction of a matrix Z to its $n+1$ columns and $n+1$ rows. Ortiz and Samara [4] have shown that the solution of the Tau problem (2), associated with (1), is equivalent to the simultaneous solution of an homogeneous problem defined by (3)–(4) for the coefficient vector $\underline{\underline{c}}_n$ of $y_n(x)$. Since the solution of the Tau problem is a polynomial of degree n, we can write

$$\underline{\underline{c}}_n[G_\lambda]_n = \underline{\underline{0}}, \tag{5}$$

where G_λ stands for the combined matrix $(VB, \hat{\Pi}_\lambda)$, the first w columns of which are those of the matrix VB, followed by those of $\hat{\Pi}_\lambda$, we can write (5) as a generalized eigenvalue problem of the form

$$A_1\underline{b} = \lambda A_2\underline{b},$$

which is the problem to be solved by using standard software. The spectral parameter λ may enter the boundary conditions, in which case the matrix B shall depend on λ.

3. Nonlinear Eigenvalue Problems

Let us consider differential eigenvalue problems of the form

$$\begin{aligned} D_\lambda y(x) := \left[\sum_{i=0}^{r} \lambda^i D_i\right] y(x) = 0, \qquad & 0 \leqslant x \leqslant b, \\ (f_j, y) = 0, \qquad & j = 1(1)\, w, \end{aligned} \tag{6}$$

where $D_i \in \mathbf{D}$, for $i = 0(1)\, r$, $r \geqslant 2$. In them the spectral parameter enters nonlinearly in the differential equation.

As before, we reduce this problem to an algebraic eigenvalue problem, which now is of the form

$$\underline{\underline{\mathbf{a}}}_n[G_\lambda]_n = \underline{\underline{\mathbf{0}}} \tag{7}$$

with

$$[G_\lambda]_n = \sum_{i=0}^{r} \lambda^i [G_i]_n, \; r \geqslant 2.$$

Transposing the matrix equation (7) we obtain the nonlinear matrix eigenvalue problem

$$\left[\sum_{i=0}^{r} \lambda^i \Gamma_i\right] \underline{\mathbf{b}} = \underline{\mathbf{0}}, \qquad r \geqslant 2, \tag{8}$$

where

$$\underline{\mathbf{b}} = [\underline{\mathbf{c}}_n]^T \qquad \text{and} \qquad \Gamma_i = [G_i]_n^T,\ i = 0(1)\,r.$$

Peters and Wilkinson [6] have remarked that the generalized eigenvalue problem (8) has the same and only the same eigenvalues as the *standard* eigenvalue problem

$$P\underline{z} = \lambda Q\underline{z}, \tag{9}$$

where

$$P := \begin{vmatrix} 0 & I & & & \\ 0 & 0 & I & & \\ \cdots & \cdots & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & 0 & I \\ \Gamma_0 & \Gamma_1 & \cdots & \cdots & \Gamma_{r-1} \end{vmatrix}, \qquad Q := \begin{vmatrix} I & & & & \\ 0 & I & & & \\ \cdots & \cdots & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & I & \\ 0 & \cdots & \cdots & 0 & -\Gamma_r \end{vmatrix}$$

and

$$\underline{z} := (\underline{\mathbf{b}}, \lambda\underline{\mathbf{b}}, \lambda^2\underline{\mathbf{b}}, \lambda^3\underline{\mathbf{b}}, ..., \lambda^{r-1}\underline{\mathbf{b}})^T \in \mathbf{R}^{r(n+1)}.$$

Peters and Wilkinson have also remarked in [6] that when Γ_r is nonsingular, the eigenvalues of the previous problems are also those of the standard eigenvalue problem

$$A\underline{z} = \lambda I\underline{z}, \tag{10}$$

where

$$A := \begin{vmatrix} 0 & I & & & \\ 0 & 0 & I & & \\ \cdots & \cdots & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & 0 & I \\ \theta_0 & \theta_1 & \cdots & \cdots & \theta_{r-1} \end{vmatrix},$$

and

$$\theta_i := -[\Gamma_r]^{-1}\,\Gamma_i, \qquad \text{for} \quad i = 0(1)\,r-1.$$

Let us assume that Γ_r is singular, but that, say, Γ_0 is not; then we may reformulate problem (8) as

$$\sum_{i=0}^{r} [\lambda^i/\lambda^r]\,\Gamma_i\underline{\mathbf{b}} = \underline{\mathbf{0}}, \qquad \text{with } r \geqslant 2,$$

and obtain a problem for $1/\lambda$.

4. Numerical Examples of Nonlinear Eigenvalue Problems

We shall consider now three numerical examples where the spectral parameter appears nonlinearly in the differential equation.

Example I. Let

$$y''(x) + \{\lambda + \lambda^2 x^2]\, y(x) = 0, \qquad -1 \leqslant x \leqslant 1, \tag{11}$$

with the boundary conditions

$$y(-1) = y(1) = 0.$$

This problem has an infinite number of negative and positive eigenvalues. Collatz (see [15, pp. 210 and 425]) estimated two eigenvalues of it by using a power series expansion: a positive one, 1.9517, and a negative one, -6.5. The same problem was considered by Chaves and Ortiz [1] by using the recursive formulation of the Tau method of Ortiz [2]; with estimations up to degree eight, they found the first to $5D$ and gave estimates of three other eigenvalues, two of them negative. More recently, Scott [16] discussed this problem by using invariant imbedding techniques which convert it into an initial value problem. His estimates were obtained by using the Runge–Kutta–Fehlberg scheme with a tolerance of 10^{-8}, coupled with a root solver to find the eigenvalues. He gives approximations of the first three positive eigenvalues. Table I displays the values obtained with the formulation of the Tau method discussed in this paper for $n = 10$, 15 for the first two positive and the first two *negative* eigenvalues. The computation is repeated again for $n = 20$ to test the accuracy of our results. It shows that for $n = 10$ the first seven digits are accurate for the first positive eigenvalue, five for the second one and four for the other two. For $n = 15$ the first ten digits of the first positive eigenvalue are accurate, nine of the second positive and seven of the first and second negative eigenvalues. The third positive eigenvalue was also computed with $n = 20$ to compare it with the result

TABLE I

Tau Method Approximation of the Eigenvalues of $y''(x) + [\lambda + \lambda^2 x^2]\, y(x) = 0$, $y(-1) = y(1) = 0$

Eigenvalues	Degree of Tau approximation		
	$n = 10$	$n = 15$	$n = 20$
λ_1	1.951 702	1.951 702 365	1.951 702 365
λ_2	4.286 1	4.286 111 06	4.286 111 061
λ_3			7.545 920 36
λ_{-1}	−6.597 1	−6.597 162	−6.597 162 005
λ_{-2}	−7.036 5	−7.035 688	−7.035 687 97
λ_{-3}			−13.200 062

reported by Scott. We find that he correctly claimed his results to be accurate up to eight digits; we have also computed the third negative eigenvalue for $n=20$.

EXAMPLE II. Let

$$y''''(x)+\lambda[y''(x)-y(x)]+\lambda^2 y(x)=0, \qquad 0\leqslant x\leqslant 1, \tag{12}$$

with the boundary conditions

$$y(0)=0; \qquad y'(0)+y'''(0)=0; \qquad y(1)=0;$$
$$y'(1)+y'''(1)=0.$$

In Table II we report numerical results obtained with the Tau method formulation of this paper for $n=10$ and $n=15$.

EXAMPLE III. In a recent paper Spence [17] considered a boundary value problem for the biharmonic equation

$$\nabla^4 u(x, y)=0, \tag{13}$$

defined on a semi-infinite strip $S := \{x\geqslant 0, -1\leqslant y\leqslant 1\}$.

The boundary conditions on the free edges $y=\pm 1$ are homogeneous. Data on the segment $x=0$ are of a type that arises both in elasticity theory and in Stokes' flow of a viscous fluid.

The solution $u(x, y)$ is assumed to be given by an expansion

$$u(x, y)=\sum_k c_k h_k(y, \lambda_k)\exp(-\lambda_k x)$$

in terms of eigenfunctions $h_k(y, \lambda_k)$, where λ_k are the eigenvalues and c_k coefficients fixed by the data on the edge $x=0$ of S. The eigenvalues λ_k are assumed to be such that $\mathrm{Re}(\lambda_k)>0$, to be compatible with the boundary conditions.

TABLE II

Tau Method Approximation of the Eigenvalues of $y''''(x)+\lambda[y''(x)-y(x)]+\lambda^2 y(x)=0$, $0\leqslant x\leqslant 1$, with the Boundary Conditions $y(0)=0$; $y'(0)+y'''(0)=0$; $y(1)=0$; $y'(1)+y'''(1)=0$

	Degree of Tau approximation	
Eigenvalues	$n=10$	$n=15$
λ_1	1.000 000	1.000 000
λ_2	12.693 84	12.693 835
λ_3	14.324 5	14.324 628
λ_4	50.51	50.532

We remark that the eigenvalues of problem (13) are implicitly defined by the differential eigenvalue problem

$$h''''(y)+2\lambda^2 h''(y)+\lambda^4 h(y)=0, \qquad -1\leqslant y\leqslant 1, \tag{14}$$

with the boundary conditions

$$h(\pm 1)=h'(\pm 1)=0,$$

and the supplementary requirement that $\mathrm{Re}(\lambda_k)>0$. These eigenvalues are the zeros with positive real parts of the transcendental equation

$$2\lambda\pm\sin(2\lambda)=0,$$

where the even (odd) eigenfunctions correspond to the + (−) sign, respectively. These eigenfunctions can be separated by solving individually the two differential eigenvalue problems

$$h''''(y)+2\lambda^2 h''(y)+\lambda^4 h(y)=0, \qquad -1\leqslant y\leqslant 1,$$

with the boundary conditions

$$h'(0)=h'''(0)=h(1)=h'(1)=0, \tag{15a}$$

for the even eigenmodes; or

$$h(0)=h''(0)=h(1)=h'(1)=0, \tag{15b}$$

for the odd eigenmodes.

We have solved the differential eigenvalue problems (15a)–(15b) with the formulation of the Tau method of this paper, taking $n=20$. In Table III we report our direct results, which *ignore* the known form of the solution, and also those given by

TABLE III

Tau Method Approximation of the Eigenvalues of a Biharmonic Problem Defined on an Infinite Strip, Equivalent to the Problem $h''''(y)+2\lambda^2 h''(y)+\lambda^4 h(y)=0$, $-1\leqslant y\leqslant 1$, $h'(0)=h'''(0)=h(1)=h'(1)=0$, and $h(0)=h''(0)=h(1)=h'(1)=0$

Exact eigenvalues		
Spence [17]	Hillman and Salzer [18]	Tau method approximate eigenvalues
2.106 196 + *i* 1.125 364		2.106 197 + *i* 1.125 364
	3.748 838 + *i* 1.384 339	3.748 839 + *i* 1.384 339
5.356 269 + *i* 1.551 574		5.356 269 + *i* 1.551 574
	6.949 980 + *i* 1.676 105	6.949 980 + *i* 1.676 105
8.536 682 + *i* 1.775 544		8.536 685 + *i* 1.775 524

Spence in [17]. The odd eigenvalues were not reported in Spence's analysis; however, they were accurately computed by Hillman and Salzer [18] in 1943 on the basis of an approximation given by G. H. Hardy. For the first four eigenvalues our results agree to at least six digits with the exact ones; for the fifth one our agreement is of at least five digits.

5. Steps in the Solution of a Concrete Example

We shall give in this section a complete description of the steps required to compute one of the examples considered in this paper. Software for the Tau method applied to the numerical solution of differential eigenvalue problems where the parameter enters nonlinearly follows essentially the same steps.

Let us consider Example I, problem (11), where the differential equation is

$$y''(x) + \{\lambda + \lambda^2 x^2\}\, y(x) = 0, \qquad -1 \leqslant x \leqslant +1. \tag{16}$$

The matrix Π_λ associated with it is

$$\Pi_\lambda := \eta^2 + \lambda I + \lambda^2 \mu^2, \tag{17}$$

where

$$\eta^2 = \left|\begin{array}{cccccc} 0 & & & & & \\ 0 & 0 & & & & \\ 2 & 0 & 0 & & & \\ & 6 & 0 & 0 & & \\ & & 12 & 0 & 0 & \\ & & & 20 & 0 & 0 \\ & & & & \cdots & \cdots \end{array}\right|;$$

$$\mu^2 = \left|\begin{array}{cccccc} 0 & 0 & 1 & & & \\ & 0 & 0 & 1 & & \\ & & 0 & 0 & 1 & \\ & & & 0 & 0 & 1 \\ & & & & \cdots & \cdots \end{array}\right|,$$

and I in the unit matrix.

The conjugate of Π_λ under the similarity transformation defined by V is given by

$$\hat{\Pi}_\lambda := V \Pi_\lambda V^{-1} = \hat{\Pi}_0 + \lambda \hat{\Pi}_1 + \lambda^2 \hat{\Pi}_2,$$

where

$$V := \left|\begin{array}{cccccc} 1 & & & & & \\ 0 & 1 & & & & \\ -1 & 0 & 2 & & & \\ 0 & -3 & 0 & 4 & & \\ 1 & 0 & -8 & 0 & 8 & \\ 0 & 5 & 0 & -20 & 0 & 16 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right|$$

is the Chebyshev matrix for $-1 \leqslant x \leqslant +1$;

$$\hat{\Pi}_0 := V\eta^2 V^{-1} = \left|\begin{array}{cccccc} 0 & & & & & \\ 0 & 0 & & & & \\ 4 & 0 & 0 & & & \\ 0 & 24 & 0 & 0 & & \\ 32 & 0 & 48 & 0 & 0 & \\ 0 & 120 & 0 & 80 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right|;$$

$$\hat{\Pi}_1 := VIV^{-1} = I;$$

and

$$\hat{\Pi}_2 := V\mu^2 V^{-1} = \left|\begin{array}{cccccc} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ & & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ & & & \frac{1}{4} & 0 & \frac{1}{2} \\ & & & \cdots & \cdots & \cdots \end{array}\right|.$$

The two boundary conditions

$$y(-1) = y(1) = 0 \tag{18}$$

are represented by the two column vectors

$$(1, -1, 1, -1, ...)^T \qquad \text{and} \qquad (1, 1, 1, 1, ...)^T, \tag{19}$$

respectively. Let us consider a concrete case, say $n = 5$,

$$[G_\lambda]_5 := [G_0]_5 + \lambda[G_1]_5 + \lambda^2[G_2]_5, \tag{20}$$

where

$$[G_0]_5 = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 4 & 0 & 0 & 0 \\ -1 & 1 & 0 & 24 & 0 & 0 \\ 1 & 1 & 32 & 0 & 48 & 0 \\ -1 & 1 & 0 & 120 & 0 & 80 \end{vmatrix};$$

$$[G_1]_5 := \begin{vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix};$$

$$[G_2]_5 := \begin{vmatrix} 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{vmatrix}.$$

Solving the algebraic eigenvalue problem defined by (20) (see (7)) we find the first two Tau method approximate eigenvalues

$$\lambda_1 \doteqdot 1.92 \qquad \text{and} \qquad \lambda_2 \doteqdot 4.19.$$

6. Final Remarks

The results presented in this paper seem to indicate that the new formulations of the Tau method considered here deserve to be taken into account for the numerical treatment of differential eigenvalue problems where the spectral parameter enters nonlinearly.

References

1. T. Chaves and E. L. Ortiz, *Z. Angew. Math. Mech.* **48**, 415 (1968).
2. E. L. Ortiz, *SIAM J. Numer. Anal.* **6**, 480 (1969).
3. E. L. Ortiz and H. Samara, *Computing* **27**, 15 (1981).
4. E. L. Ortiz and H. Samara, *Computing* **31**, 95 (1983).
5. C. Lanczos, *J. Math. Phys.* **17**, 123 (1938).

6. G. Peters and J. H. Wilkinson, *SIAM J. Numer. Anal.* **7**, 479 (1970).
7. K. M. Liu and E. L. Ortiz, Numerical approximation of differential eigenvalue problems with discontinuous coefficients, Imperial College, Res. Rep. NAS 0885, (1985), unpublished.
8. E. L. Ortiz, in *Numerical Treatment of Differential Equations in Applications*, edited by R. Ansorge and W. Tornig (Springer-Verlag, Berlin, 1978), p. 127.
9. E. L. Ortiz and H. Samara, *Comput. Math. Appl.* **10** 5 (1984).
10. P. Onumanyi and E. L. Ortiz, *Math. Comput.* **43**, 189 (1984).
11. E. L. Ortiz and A. Pham Ngoc Dinh, *Nonlinear Anal.* **9**, 53 (1985).
12. E. L. Ortiz and K. S. Pun, *J. Comput. Appl. Math.* **12**, **13**, 511 (1985).
13. E. L. Ortiz and K. S. Pun, *Comput. Math. Appl. B* **12**, 1225 (1986).
14. S. Namasivayam and E. L. Ortiz, in *Numerical Treatment of Differential Equations*, edited by K. Strehmel (Teubner, Leipzig), in press.
15. L. Collatz, *Eigenwertaufgaben mit Technischen Anwendungen* (Akademische Verlagsgesellschaft; Geest & Portig K.-G., Leipzig, 2nd. ed., 1963).
16. M. R. Scott, in *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, edited by A. K. Aziz (Academic Press, New York, 1975), p. 89.
17. D. A. Spence, *Inst. Math. Appl., J. Appl. Math.* **30**, 107 (1983).
18. P. Hillman and H. E. Salzer, *Philos. Mag.* **34**, No. 7, 575 (1943).